§1 General.

X = top. space, A = comm. ring, $\mathcal{F} = \mathrm{Shv}_{A\text{-Mod}}(X)$.

$\mathcal{U} = \{U_i \mid i \in I\}$ open cover of X.

$i_0, \dots, i_p \in I$, $U_{i_0 i_1 \cdots i_p} := U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}$.

Defn. $C^\cdot(\mathcal{U}, \mathcal{F})$ defined by:

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{(i_0, \dots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0 \cdots i_p}).$$

for $f \in C^p(\mathcal{U}, \mathcal{F})$, $df \in C^{p+1}(\mathcal{U}, \mathcal{F})$ given by

$$(df)_{i_0 \cdots i_{p+1}} := \sum_{k=0}^{p+1} (-1)^k f_{i_0 \cdots \hat{i_k} \cdots i_{p+1}} \Big|_{U_{i_0 \cdots i_{p+1}}}.$$

e.g. $C^0 \to C^1 \to C^2 \to \cdots$

$$\prod_{i \in I} \mathcal{F}(U_i) \longrightarrow \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j) \longrightarrow \prod_{(i_0, i_1, i_2) \in I^3} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap U_{i_2}).$$

$$(f_i)_{i \in I} \longmapsto \left(\mathrm{res}^{U_j}_{U_i \cap U_j}(f_j) - \mathrm{res}^{U_i}_{U_i \cap U_j}(f_i)\right).$$

$$(f_{ij}) \longmapsto f_{i_1 i_2}\Big|_{U_{i_0 i_1 i_2}} - f_{i_0 i_2}\Big|_{U_{i_0 i_1 i_2}} + f_{i_0 i_1}\Big|_{U_{i_0 i_1 i_2}}.$$

Exercise: $d^2 = 0$.

Defn. $f \in C^p(\mathcal{U}, \mathcal{F})$ is <u>alternating</u> if

(1) $f_{i_0 \cdots i_p} = 0$ if ≥ 2 indices same.

(2) $f_{\sigma(i_0)\sigma(i_1)\cdots\sigma(i_p)} = \mathrm{sgn}(\sigma) \cdot f_{i_0 \cdots i_p}$ for every $\sigma \in S_{p+1}$.

Exercise: d(alternating) is alternating.

$$i : C'(\mathcal{U}, \mathcal{F}) \hookrightarrow C(\mathcal{U}, \mathcal{F}) \quad \text{sub-cplx of alternating cochains.}$$

Defn. If I is endowed with a ~~total~~ ordering "$<$".

$$\begin{tikzcd} \prod\limits_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0 \dots i_p}) =: C''(\mathcal{U}, \mathcal{F}) \arrow[r, hook, shift right] & C(\mathcal{U}, \mathcal{F}) \arrow[l, two heads, shift right] \end{tikzcd}$$

direct summand

Exercise: (1) d on $C(\mathcal{U}, \mathcal{F})$ restricts to a diff. d on $C''(\mathcal{U}, \mathcal{F})$.

(2)
$$\begin{tikzcd} C' \arrow[r, hook] \arrow[dr, "\cong"', "\text{isom of cplxes}"] & C \arrow[d] \\ & C''. \end{tikzcd}$$

(3)
$$\begin{tikzcd} & H^0(C'(\mathcal{U}, \mathcal{F})) \arrow[r, "\cong"] & H^0(C(\mathcal{U}, \mathcal{F})) \arrow[d, "\cong"] \\ \mathcal{F}(X) \arrow[ur, "\cong"] \arrow[urr, "\cong"] \arrow[rr, "\cong"] & & H^0(C''(\mathcal{U}, \mathcal{F})) \end{tikzcd}$$

Fact: $C \twoheadrightarrow C'' \cong C' \hookrightarrow C$ is homotopic to identity.

$$\Rightarrow \quad H^i(C'(\mathcal{U}, \mathcal{F})) \xrightarrow{\cong} H^i(C(\mathcal{U}, \mathcal{F})) \xrightarrow{\cong} H^i(C''(\mathcal{U}, \mathcal{F})).$$

Defn. $\mathcal{V} = \{V_j \subseteq X\}_{j \in J}$ is a refinement of $\mathcal{U}$ if $\exists\, \sigma : J \to I$,

s.t. $V_j \subseteq U_{\sigma(j)} \quad \forall j \in J$.

Such σ induces $\sigma^* : C(\mathcal{U}, \mathcal{F}) \longrightarrow C(\mathcal{V}, \mathcal{F})$.

Fact: σ^* is independent of choice of σ.

$\{X\} \prec \{X, U_i\} \prec \{X\} \Rightarrow H^{>0}(C(\mathcal{U}, \mathcal{F})) = 0$ if $X \in \mathcal{U}$.

Defn. X = top. space, $\mathcal{F} \in \mathrm{Shv}_{A\text{-mod}}(X)$.

$$\check{H}^i(X, \mathcal{F}) := \operatorname*{colim}_{\substack{\cdot\ \mathcal{U} \text{ open cov. of } X \\ \cdot\ \text{refinement}}} H^i(\mathcal{U}, \mathcal{F}).$$

Thm (Leray acyclicity thm). $\mathcal{F} \in \mathrm{Shv}_{A\text{-mod}}(X)$. $\underset{\{U_i\}_{i \in I}}{\overset{\parallel}{\mathcal{U}}}$ = open cover of X.

Suppose $\exists$ family of coverings $\{\mathcal{V}^\lambda\}_{\lambda \in \Lambda}$, $\mathcal{V}^\lambda = \{V_j^\lambda\}_{j \in J_\lambda}$, cofinal under refinement,

s.t. $H^{>0}\left(\mathcal{V}^\lambda\big|_{U_{i_0 \cdots i_p}}, \mathcal{F}\big|_{U_{i_0 \cdots i_p}}\right) = 0 \quad \forall (i_0, \cdots, i_p) \in I^{p+1}$

Then the canonical map $H^*(\mathcal{U}, \mathcal{F}) \xrightarrow{\cong} \check{H}(X, \mathcal{F})$.

pf: double cplx $\displaystyle\prod_{J^{p+1} \times I^{q+1}} \mathcal{F}(V^\lambda_{j_0 \cdots j_p} \cap U_{i_0 \cdots i_q})$.

$\rightarrow$ first, then $\uparrow$: $H^*(\mathcal{U}, \mathcal{F})$ by condition

$\uparrow$ first, then $\rightarrow$: will be $H^*(\mathcal{V}^\lambda, \mathcal{F})$ as soon as $\mathcal{V}^\lambda$ refinement of $\mathcal{U}$.

§2. QCoh / sep. schemes.

Goal: Show that

- $X =$ sep. scheme $/\mathrm{Spec}(A)$, $\mathcal{F} \in \mathrm{QCoh}(X) \overset{\text{faithful}}{\subseteq} \mathrm{Shv}_{A\text{-mod}}(X)$.
- $\mathcal{U} =$ any open cover by affines in X
- $\{\mathcal{V}^\lambda\}_{\lambda\in\Lambda} =$ family of "all" open cover by affines in X.

satisfies the above assumption.

Prop. $X =$ affine, $= \mathrm{Spec}(A)$, $\mathcal{V} = \{V_j\}_{j\in J}$, $\overset{\smile}{=} \mathrm{Spec}(A_j)$, is a covering by affine opens, $\underset{\tilde{M}}{\mathcal{F}} \in \mathrm{QCoh}(X)$. ($M$ A-mod.)

Then $$H^i(\mathcal{V}, \mathcal{F}) = \begin{cases} M & i = 0 \\ 0 & i > 0 \end{cases}.$$

pf. Case I: J is finite

$$M \to \prod_{j\in J} M \otimes_A A_j \longrightarrow \prod_{(j_1,j_2)\in J^2} M \otimes_A A_{j_1} \otimes_A A_{j_2} \to \cdots$$

is exact:
- take stalk
- fin. prod comm w/ filtered colim
- $\forall \mathfrak{p} \subseteq A$, $A_\mathfrak{p} \cong (A_j)_\mathfrak{p}$ for some j, so Cech coh van. above deg 0.

Case II, general case.

Hint: choose $J' \subseteq J$ finite which already cover.

form double cpx.

use case I for 1 direction & use $V_{j'} \in \{\mathcal{V}\}$ for the other.

Cor. $X =$ sep. schm, $\mathcal{U} =$ affine open cover, $\mathcal{F} \in QCoh(X)$, then

$$H^*(\mathcal{U}, \mathcal{F}) \xrightarrow{\cong} \check{H}^*(X, \mathcal{F}).$$

LHS doesn't depend on choice of $\mathcal{U}$!

Cor. $\begin{array}{c} X \\ f \downarrow \\ Y \end{array}$ qc, sep., $\mathcal{F} \in QCoh(X)$, the assignment

$$\mathrm{Spec}(A) \underset{\text{open}}{\subseteq} Y \mapsto \check{H}^i(X_A, \mathcal{F}) \text{ defines a}$$

qcoh sheaf on Y.

Defn. Denoted as $R^i f_* \mathcal{F}(\mathrm{Spec}(A)) := \check{H}^i(X_A, \mathcal{F})$.

Rmk. • $R^0 f_* \mathcal{F} = f_* \mathcal{F}$.

• $\check{H}^*(\mathbb{P}^n_A, \mathcal{F}) =$ coh of. $\left(\bigoplus_{i=0}^{n} \mathcal{F}(D_+(x_i)) \to \bigoplus_{i<j} \mathcal{F}(D_+(x_i x_j)) \to \cdots \right)$

which is what was defined before.

• $\left.\begin{array}{l} X \xrightarrow{f} Y \text{ affine} \\ \mathcal{F} \in QCoh(X) \end{array}\right\} \Rightarrow R^i f_* \mathcal{F} = \begin{cases} f_* \mathcal{F} & i = 0 \\ 0 & \text{o/w.} \end{cases}$

• $X \xrightarrow[\text{affine}]{f} Y \xrightarrow[g]{qc, sep} Z$, $h = g \circ f$ $\Rightarrow R^i h_* \mathcal{F} = R^i g_* (f_* \mathcal{F})$.

• $X \xrightarrow{f} Y$ qc, sep., $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ exact seq. in $QCoh(X)$

$\rightsquigarrow$ LES: $0 \to f_* \mathcal{F}_1 \to f_* \mathcal{F}_2 \to f_* \mathcal{F}_3 \to R^1 f_* \mathcal{F}_1 \to R^1 f_* \mathcal{F}_2 \to \cdots$

• Can define $H^*(X, \mathcal{F})$ in a different way (derived functor),

(i) agree w/ $\check{H}^*(X, \mathcal{F})$ when $X =$ qc, sep, $\mathcal{F} \in QCoh(X)$.

(ii) $H^{\leq 1} = \check{H}^{\leq 1}$ always.

Thm (Serre's criterion) X = qc sep. schm. TFAE:

(1) X affine,

(2) $\forall \mathcal{F} \in QCoh(X)$, $\check{H}^{>0}(X, \mathcal{F}) = 0$,

(3) $\forall \mathcal{I}$ qcoh ideal sheaf, $\check{H}^1(X, \mathcal{I}) = 0$.

pf shows that we only need X qc qs.

pf: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) obvious.

(3) $\Rightarrow$ (1): $A := \mathcal{O}_X(X)$. $X \longrightarrow \mathrm{Spec}(A)$.

Claim: $\forall x \in X$, $\exists f \in A$, s.t. $x \in X_f := X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_f)$.
$A = \Gamma(X, \mathcal{O}_X)$ & X_f affine.

pf: X qc $\Rightarrow \overline{\{x\}} \subseteq X$ qc $\Rightarrow \exists$ closed pt $x' \in \overline{\{x\}}$. So suffices to show when $x \in X^{cl}$.

Now $\{x\}_{red} = V(\mathcal{M})$.

& $x \in U \overset{\text{affine open}}{\subseteq} X$, so $(X \setminus U)_{red} = V(\mathcal{J})$.

$$0 \to \mathcal{M}\cdot\mathcal{J} \to \mathcal{J} \to \mathcal{J}/\mathcal{M}\mathcal{J} \to 0$$

$\mathcal{J}/\mathcal{M}\mathcal{J}$ is $k(x)$ supp. @ x.

$\Gamma(X, \mathcal{O}_X) \supseteq \Gamma(X, \mathcal{J}) \ni f \longmapsto 1 \in k(x)$ supp. @ x.

Now $f_x \notin m_x \cdot \mathcal{O}_{X,x} \rightsquigarrow x \in X_f = D(f|_U)$ affine. $\square$

Finitely many f_i's will have $X = \bigcup_{i=1}^r X_{f_i}$.

We've showed $\mathcal{O}(X_{f_i}) = A_{f_i}$ (only need X qc qs!)

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d] & X \arrow[d, "\cong"] \\ \mathrm{Spec}(A) & \bigcup_{i=1}^r \mathrm{Spec}(A_{f_i}) \arrow[l, phantom, "\supseteq"] \end{tikzcd}$$

Remains to show: $(f_i) = A$.

To that end: consider $\varphi: \mathcal{O}_X^{\oplus r} \xrightarrow{(f_1, \cdots, f_r)} \mathcal{O}_X$.

H^1(qcoh ideal sheaf) $= 0 \Rightarrow H^1(\ker(\varphi)) = 0$

$\Rightarrow A^{\oplus r} \xrightarrow{(f_1, \cdots, f_r)} A$ $\square$.